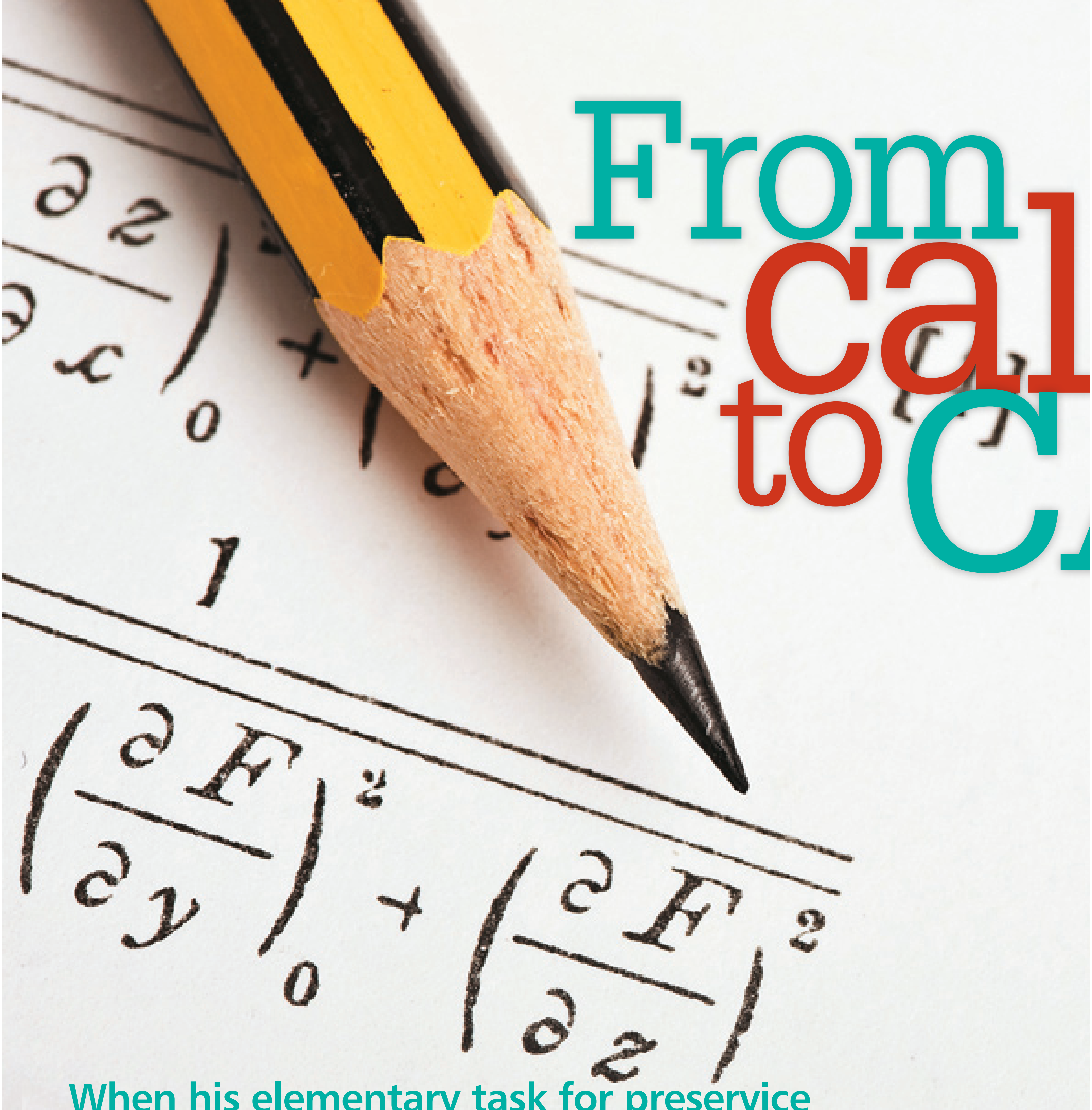
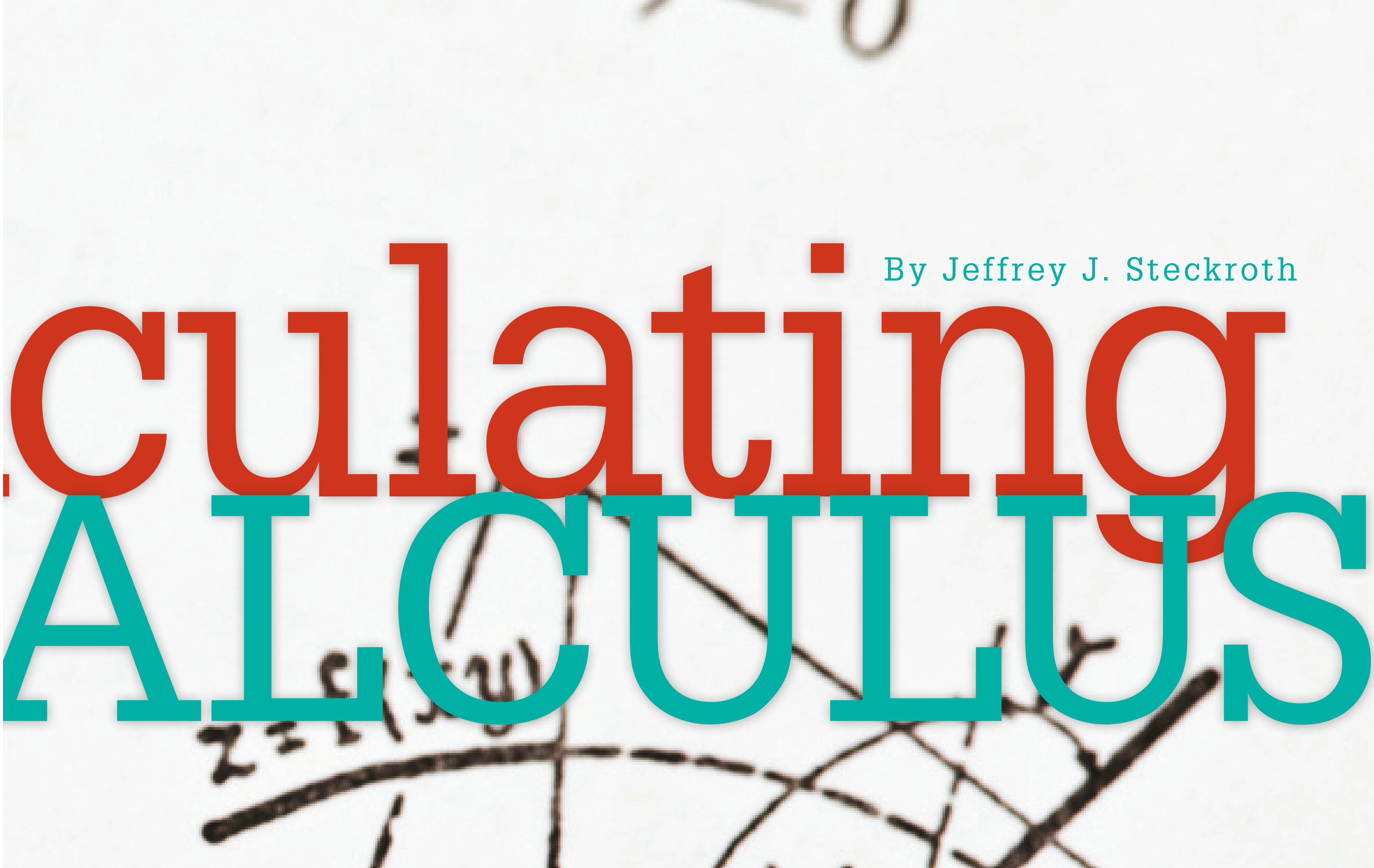


# From cal to C

**When his elementary task for preservice teachers became an excursion across grade levels, this former secondary school teacher gained appreciation for how closely connected elementary school multiplication is to the algebraic thinking of secondary school mathematics.**

# culating ALCULUS

By Jeffrey J. Steckroth



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For nearly three decades, during which I taught everything from basic algebra to advanced placement calculus, I thought of myself as a secondary school mathematics teacher. The notion of teaching elementary school math never appealed to me because of its simplicity. Surely anyone could teach children to count, add, subtract, multiply, divide, and work with fractions, decimals, and percents. Teaching high school math is a much more challenging task that only serious mathematicians (like myself) can handle.

My self-concept began to change when I started teaching an elementary school mathematics methods course to preservice teachers at Old Dominion University, where I am an assistant professor of mathematics education. Although I had received a healthy dose of elementary school math education when I studied secondary school mathematics education as a doctoral student at the University of Virginia, I had doubted that I would ever use that part of my education. I found my studies interesting but did not believe they were relevant for me—until years later when I learned that I would be teaching the elementary school methods course.

As the semester unfolded, again and again I realized just how fundamental the mathematics in the elementary grades is for success at higher levels. Number sense, pattern recognition, problem-solving ability, and a positive disposition toward math begin long before students enter middle or high school. Activities that seem routine at first have a way of transforming into rich, multilayered tasks with broad appeal and bountiful connections among mathematical topics. One episode in particular, involving the creation of a multiplication table, led my methods class and me on a mathematical excursion across grade levels.

## Discovery of "hidden" challenges

One day I asked my preservice college students to fill in a blank multiplication table so that we could discuss different ways of doing so and then see what patterns we would recognize. Several students described filling in by rows, some by columns, and others by actually calculating each product. Once the table was complete, we talked about the multiplicative property of zero and the fact that the number one is the identity element for multiplication. We then discussed the types of numerical patterns that students had identified in the rows, columns, and diagonals.

The National Council of Teachers of Mathematics (NCTM) Illuminations Web site (http://illuminations.nctm.org/) has many interesting activities, one of which, called "Times Tables," is designed for use with students in grades pre-K–2. We went to the Web site and verified that the contents of the online table were the same as those that my students had filled in. I then reset the table so that no products were exposed and clicked on the cells (see **fig. 1**) to reveal a set of nine products for everyone to examine. I asked students to study the numbers in the diagonal to see if they could find any evidence of a pattern. This task proved to be a challenge, but after some thought, several observations emerged.

FIGURE 1

The instructor used NCTM's Illuminations Web site to reveal a set of nine products on a multiplication table and asked students to study the numbers in the diagonal to see if they could find any evidence of a pattern.

**Times Table**

| x | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | | | | | | | | | 0 | |
| 1 | | | | | | | | 7 | | |
| 2 | | | | | | | 12 | | | |
| 3 | | | | | | 15 | | | | |
| 4 | | | | | 16 | | | | | |
| 5 | | | | 15 | | | | | | |
| 6 | | | 12 | | | | | | | |
| 7 | | 7 | | | | | | | | |
| 8 | 0 | | | | | | | | | |
| 9 | | | | | | | | | | |

## Symmetry

The first observation was that the highlighted values have symmetry. Numbers to the right and above sixteen are identical to those to the left and below sixteen. I asked why this is the case, and someone noted that the products for the number pairs are the same except for order. For example, 5 × 3 equals 15, and 3 × 5 also equals 15, evidence of the commutative property of multiplication.

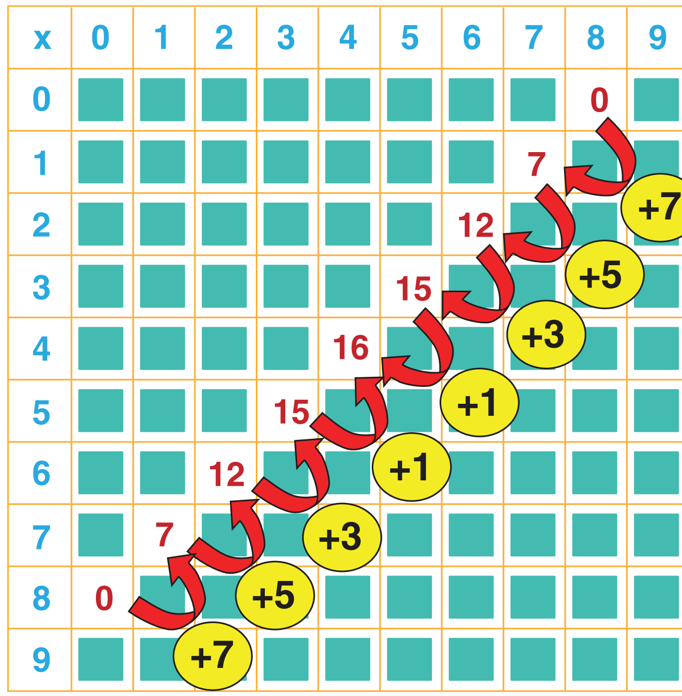
## Successive increases

The second explanation focused on successive differences between values on the diagonal, which are consecutive odd numbers. For instance, starting in the upper right-hand corner and moving from zero to seven, the increase is seven; the next increase, from seven to twelve, is five; from twelve to fifteen, the increase is three, and from fifteen to sixteen, the increase is one. Moving from the lower left-hand corner of the table toward the middle value of sixteen, the same successive increases appeared (see **fig. 2**). The discussion went no further, so I asked the class to tell me some way in which all of the highlighted values in the table are alike.

When this question stumped students, I asked them to focus on the two factors—one in the top row and one in the first column—that are paired with each number on the diagonal.

**FIGURE 2**

After two observations—symmetry as evidence of the commutative property of multiplication and successive differences between values on the diagonal resulting in consecutive odd numbers—the class discussion came to a halt.




Each such pair of numbers has something in common; I asked the class to figure it out. A long period of silence finally broke when Sarah spoke up: "They all add up to eight."

Sarah observed that the six in the first column and the two in the top row have a sum of eight and a product of twelve. I pointed out several other pairs of numbers until everyone seemed to understand that each pair does have a sum of eight, although the products are different.

Next I suggested that we organize the set of numbers into a table so that we could study and keep track of them more closely. I opened a spreadsheet (see **table 1**) and made sure that class members understood how the numbers were organized and how they connected to the values in the multiplication table. We labeled the columns *first*, *second*, and *product* on the basis of where the numbers appear in the multiplication table.

Wanting to make a connection between this set of values and something else with

**TABLE 1**

Together, the class organized a set of multiplication number pairs into a table.

| | A | B | C |
|---|---|---|---|
| 1 | first | second | product |
| 2 | 0 | 8 | 0 |
| 3 | 1 | 7 | 7 |
| 4 | 2 | 6 | 12 |
| 5 | 3 | 5 | 15 |
| 6 | 4 | 4 | 16 |
| 7 | 5 | 3 | 15 |
| 8 | 6 | 2 | 12 |
| 9 | 7 | 1 | 7 |
| 10 | 8 | 0 | 0 |

FIGURE 3

The area model was another multiplication model that the class used during the semester.

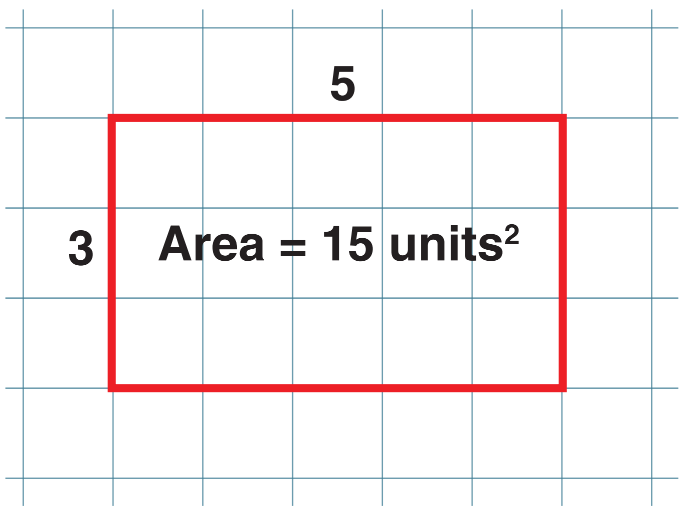



TABLE 2

Not many preservice teachers could initially make the leap to algebraic thinking.

| | A | B | C |
|---|---|---|---|
| 1 | first | second | product |
| 2 | length | width | area |
| 1 | *x* | 8 – *x* | *y* |
| 2 | 0 | 8 | 0 |
| 3 | 1 | 7 | 7 |
| 4 | 2 | 6 | 12 |
| 5 | 3 | 5 | 15 |
| 6 | 4 | 4 | 16 |
| 7 | 5 | 3 | 15 |
| 8 | 6 | 2 | 12 |
| 9 | 7 | 1 | 7 |
| 10 | 8 | 0 | 0 |

which the students were familiar, I then asked whether anyone could think of another model for multiplication that we had used this semester. When no one could remember, I drew an outline of a rectangle on a SMART Board™ grid as a hint.

Carmen quickly chimed in with "length times width equals area," so I labeled the rectangle's length and width with two numbers from our list and then showed the rectangle's area (see **fig. 3**).

I asked whether anyone could tell the class about another attribute of rectangles besides area, and Andrew mentioned perimeter. When I asked the class for the perimeter of the rectangle that we had just labeled, several students answered, "Sixteen units."

I posed another question: "What would happen if we used a different pair of numbers from our list for the length and width of the rectangle?"

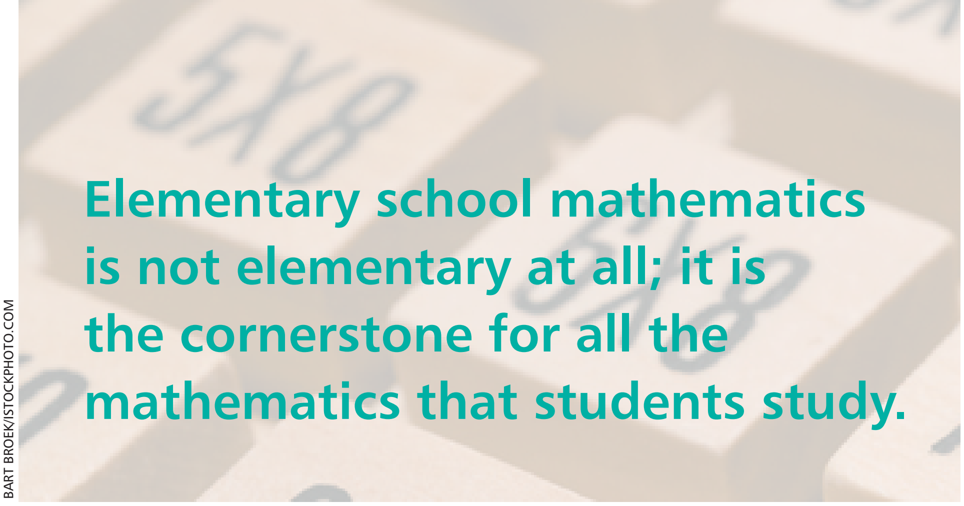
Victoria stated that the perimeter of every rectangle in our list would be the same, sixteen units, although the areas would differ. I pointed out that because the perimeter of a rectangle is made up of two lengths and two widths, the sum of the length and width of the rectangle would always be half the perimeter—in our case, eight units.

We were now using the spreadsheet data to represent a rectangle's length, width, and area, so we added another row at the top with additional labels to reflect this fact. We also talked about the first and last lines of numbers in the spreadsheet and how they are not realistic when we talk about areas because a length or width of zero units and an area of zero square units cannot exist.

Satisfied that students were aware of the geometric representation of the numbers in our lists, I asked the class to look at the numbers in the first column—length—and in the third column—area—to see if they could identify a relationship or a pattern. Jamie noted that as the length increases from one unit to four, the area increases from seven square units to sixteen. After that, the length continues to increase from five units to seven, but the area decreases from fifteen square units to seven. I was pleased at how nicely Jamie was able to articulate the relationship, and I decided that our class was ready for the next step.

## Algebraic thinking

We had reached the point in our lesson where it was time to focus on the transition from concrete representations and numerical reasoning to algebraic thinking. I asked the class if



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**Elementary school mathematics is not elementary at all; it is the cornerstone for all the mathematics that students study.**

we could generalize about the numbers in the table and look at the relationship algebraically. We had identified numerical patterns that were present in the table, so I asked if anyone could think of a way to represent the numbers in the first two columns symbolically.

One of five Content Standards (NCTM 2000), the Algebra Standard states that instructional programs at all grade levels should enable students to become proficient at several tasks:

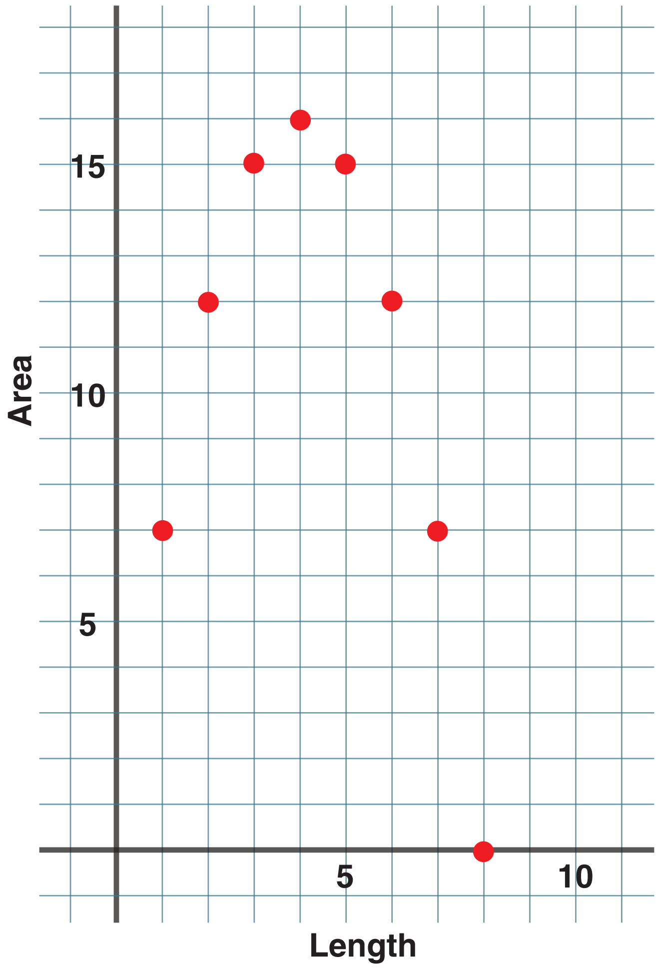
- **Understand** patterns, relations, and functions.
- **Represent** mathematical situations using algebraic symbols.
- **Use** mathematical models to represent and understand quantitative relationships.
- **Analyze** change in different contexts.

Sarah, a student with a strong background in mathematics, suggested using $x$ and $8 - x$ to represent each pair of numbers. Sensing that most of the class did not understand where $8 - x$ came from, I asked Sarah to explain how she had arrived at that expression. She correctly explained that the number in the second column, width, is always equal to the number in the first column subtracted from eight. For instance, if length equals five, then width must be eight minus five, or three. In general, if the first number is $x$, then the second number must be eight minus $x$.

I wrote these expressions in the spreadsheet under *length* and *width* (see **table 2**) and then labeled the area $y$ to distinguish it from the other values. I asked students to focus on the first column of values, $x$, and the third column

FIGURE 4

Ordered pairs produced a graph with a collection of points that connected to produce an upside-down parabola.




of values, $y$, and think about how we might represent those pairs of numbers graphically. Before long, Anne pointed out that they could be treated as ordered pairs and graphed as a series of points on a coordinate plane. I asked whether anyone could predict what such a graph might look like, but no one seemed willing to speculate.

I opened a blank page on the SMART Board, inserted a grid, and added a pair of axes. We began to plot the individual points. The ordered pairs soon produced a graph (see **fig. 4**) with an interesting collection of points that—when connected—produce "an upside down parabola," according to Jennifer. To ensure that everyone was clear about what each ordered pair represents, we discussed what the graph's point coordinates mean. The $x$-coordinate represents one factor in the multiplication table, as well as the length of a rectangle with a perimeter of sixteen units; the $y$-coordinate represents the product in the multiplication table, as well as the area of the rectangle.

Finally I asked everyone to look at the highest point on the graph and think about what its coordinates represent. All agreed that the ordered pair is (4, 16), which corresponds to a rectangle with a length of four units and an area of sixteen square units. Sarah exclaimed, "It's a square—the rectangle with the largest area is a square!" I gave everyone a chance to mull over Sarah's discovery.

## The bigger picture

The NCTM Connections Standard states that instructional programs at all grade levels should enable students to recognize connections among mathematical ideas and understand how mathematical ideas build on one another to produce a coherent whole. A lesson that had begun with a multiplication table moved into a lesson on an area model of multiplication and concluded as an algebraic model of the same situation. The graph of the parabola created an opportunity for us to pursue some higher-level mathematics: advanced algebra, precalculus, and even calculus. Mathematical topics such as the characteristics of parabolas, slopes of curves, tangents to curves, maximum values, and derivatives of functions were within our grasp.

Rather than overwhelming my students, I decided to save those topics for another day. We had already succeeded in making connections between a second-grade multiplication table and algebraic patterns—the rest could wait.

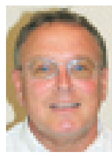
When class ended and I had a chance to reflect on the lesson, it occurred to me that this was one of those occasions when the big picture had started to come into focus. The individual mathematical concepts—multiplication, area, perimeter, patterns, and algebra—had been linked together in such a way that it was clear how they were related. My preservice teachers had been part of a lesson in which topics were not treated as isolated bits of knowledge they needed to learn, but were all part of a coherent whole—mathematics.

In the process, we had used numerical, graphical, pictorial, and algebraic representations of a function in keeping with the NCTM Representation Process Standard. We had solved problems along the way as recommended in the NCTM Problem-Solving Process Standard. Furthermore, these processes had been facilitated by the use of software such as Microsoft Office Excel® and SMART™ Notebook, plus an Internet resource, the NCTM Illuminations Web site, which had been our starting point. The NCTM Technology Process Standard was in evidence throughout this lesson.

I will not know for some time what impact this lesson had on my preservice teachers, but I do know that it had an impact on me. My first semester of teaching elementary school mathematics methods for preservice teachers has been an experience that I will not soon forget. I have learned that elementary school mathematics is not elementary at all; it is the cornerstone for all the mathematics that students study in the middle grades and high school—even calculus.

## REFERENCE


National Council of Teachers of Mathematics (NCTM). *Principles and Standards for School Mathematics*. Reston, VA: NCTM, 2000.


**Jeff Steckroth, jsteckro@odu.edu,** is an assistant professor of mathematics education at Old Dominion University in Norfolk, Virginia, where he teaches methods courses for preservice teachers and graduate courses for teachers who are working toward an endorsement as a pre-K–8 mathematics specialist. His research focuses on the use of technology to enhance student conceptual understanding of mathematics.


You will find the "Times Table" activity on NCTM's Illuminations Web site at **http://illuminations.nctm.org/ActivityDetail.aspx?ID=155**.